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On the radiative part of the Maxwell tensor for a Liénard–Wiechert field

H N Núñez-Yépez†§, J L López-Bonilla‡ and A L Salas-Brito‡‖

† Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, Apartado Postal 55-534, Iztapalapa 09340, DF México

‡ Departamento de Ciencias Básicas, Universidad Autónoma Metropolitana-Azcapotzalco, Apartado Postal 21–726, Coyoacán 04000, DF México

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Abstract. The Maxwell energy-momentum tensor associated with the Liénard–Wiechert field of a point charge in arbitrary motion splits naturally into a bounded and a radiative part. It is known that the bounded part of the Maxwell tensor does not contribute to the energy-momentum balance between matter and field which is completely accounted for by its radiative part only. In this paper we show that the purely radiative part of the Maxwell tensor can be further decomposed as the sum of a term, again not contributing to the energy-momentum balance, plus a part which is completely responsible for it. Furthermore, we also manage to find that the full radiative part of the Maxwell tensor can be generated by a superpotential that has to be regarded as non-local since its definition involves integration over a finite section of the charge's world line.

1. Introduction

A charged particle moving arbitrarily in the flat Minkowski four-space generates at points x^r on its forward light cone, see figure 1, a retarded Liénard–Wiechert electromagnetic field (LWF) [1] which in Heaviside–Lorentz units with $c = 1$ is

$$\begin{aligned} F^{ab}(x^r; z^r(\tau)) &= \frac{q}{R^2}\left(k^{[a}\ddot{z}^{b]} + \frac{(1-W)}{R}k^{[a}\dot{z}^{b]}\right) \\ &= \frac{q}{R^2}(U^a k^b - k^a U^b) \end{aligned} \tag{1}$$

where q is the particle's electric charge, $z^r(\tau)$ (or P) stands for the particle's position on its world line, C, as a function of the proper time τ, $v^b = \mathrm{d}z^b/\mathrm{d}\tau$ is the four-velocity, $a^b = \mathrm{d}v^b/\mathrm{d}\tau$ is the four-acceleration, $k^r \equiv x^r - z^r(\tau)$ so that k^r is null: $k_i k^i = 0$, $R \equiv -k^r v_r$ is the retarded distance from P to the field point Q on the null cone, $W \equiv -k^r a_r$, $U^c = Bv^c + a^c$, $B \equiv (1 - W)/R$ is the Plebański [2] invariant and $X_{[ab]} \equiv X_{ab} - X_{ba}$ stands for asymmetrization. The four-vector U^a is known as the Synge vector [3] and has some interesting properties, for example, $U^c{}_{,c} = 0$ (the Lorentz condition), $U^c R_{,c} = 0$, and $U^c k_c = -1$. It is also quite obvious that the first term on the right-hand side of the first line in equation (1) can be regarded as the radiative part of the field since its energy flux is non-vanishing even very far from the charge, whereas the second term, with an asymptotically vanishing flux, can be regarded as bounded to the moving charge [4].

§ E-mail address: nyhn@xanum.uam.mx
‖ E-mail address: asb@hp9000a1.uam.mx

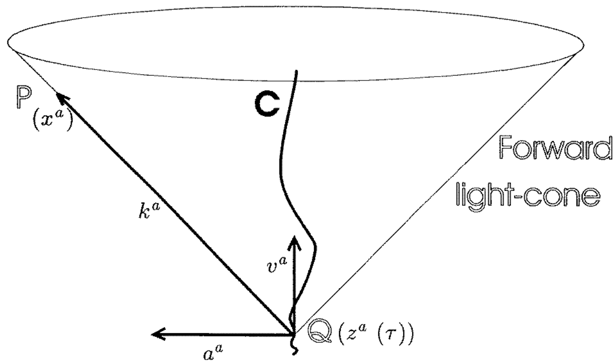

Figure 1. Kinematics of the world line, C, of the point-charge emitting the Liénard–Wiechert field. The field-point $x^r(P)$ is on the forward light cone, $z^r(\tau)(Q)$ is the charge's position on the world line, $v^b = \mathrm{d}z^b/\mathrm{d}\tau$ is the four-velocity, $a^b = \mathrm{d}v^b/\mathrm{d}\tau$ is the four-acceleration, $k^r \equiv x^r - z^r(\tau)$ so k^r is null, $R \equiv -k^r v_r$ is the retarded distance from P to Q. To write some of the quantities of interest we define the quantities $B \equiv (1 + k^r a_r)/R$ and $U^r \equiv Bv^r + a^r$.

The energy and momentum of the electromagnetic field are contained in a symmetric tensor, the energy-momentum or Maxwell tensor is defined as [1]

$$\begin{aligned} T^{ab} &\equiv F^a{}_c F^{bc} - \tfrac{1}{4}(F_{ij}F^{ij})\eta^{ab} \\ &= T_R^{ab} + T_B^{ab}. \end{aligned} \tag{2}$$

In the last line of equation (2) we have explicitly written up the splitting of the Maxwell tensor as a sum of its radiative (R) part:

$$T_R^{ab} = \frac{q^2}{R^4}\left(a^r a_r - \frac{W^2}{R^2}\right) k^a k^b \tag{3}$$

plus its bounded (B) part:

$$T_B^{ab} = \frac{q^2}{R^4}\left(\frac{1}{2}\eta^{ab} + k^{(a}a^{b)} + Bk^{(a}v^{b)} - \frac{(1-2W)}{R^2}k^a k^b\right) \tag{4}$$

in these expressions $\eta^{ab} = \mathrm{diag}(-1, 1, 1, 1)$ is the flat Minkowski metric. As is well known [1, 5–11] the energy-momentum tensor is determined by the field equations, except for the possible addition of the divergence of a tensor vanishing at spatial infinity; this is at least partly due to the fact that the only measurable quantities associated with T^{ab} are the energy and momentum fluxes and these are defined as integrals over the boundaries of any three-volume, V_3 —and so the divergence does not contribute to the fluxes. Moreover, as it must be clear, the radiative part of the energy-momentum tensor fulfills Villarroel's [6] conditions for a radiation tensor

$$T_R{}^{ib}k_b = 0 \qquad T_R{}^{ib}{}_{,b} = 0. \tag{5}$$

In this work we address two questions concerning the Maxwell tensor of a LWF. We first show, in section 2, that the non-uniqueness of the Maxwell tensor allows us to introduce a modified Maxwell tensor $\tilde{T}_{ij} \neq T_{ij}$. Both tensors, T^{ab} and $\tilde{T}^{ab}$, produce the same value for the fluxes; given this property, we claim that these two energy-momentum tensors are completely equivalent to each other—at least in those matters pertaining to a Bhabha–Synge tube [3, 10] and the energy-momentum fluxes through it. In section 3, we show that the radiative part of a Maxwell tensor can be generated by a non-local superpotential of the type first introduced by Van Weert [12], except for the non-locality of the one proposed here.

2. The splitting of the radiative part of the energy-momentum tensor

In this section we are interested only in the radiative part of a LWF, so we can, without losing any generality, assume that the whole electromagnetic field is a radiation field, i.e. we assume that T_{ab} coincides with the radiative part $T_{R\,ab}$ of the LWF [5, 13] and that the three-dimensional region mentioned above can be regarded as a Bhabha–Synge tube [13]. As we explicitly exhibit below, we can split the radiative Maxwell tensor as

$$T_{Rij} = \tilde{T}_{Rij} + A_{ij} \tag{6}$$

where A_{ij} is a tensor *not* contributing to the flux through the light cone associated with the point charge nor to the flux through a surface of constant retarded distance R [3]. We can say then that A_{ij} is the inactive part of the radiative tensor T_{Rij} since it does not participate in the energy-momentum balance through the Bhabha–Synge tube, and that $\tilde{T}_{Rij}$ is the modified Maxwell tensor for the radiation field.

To put (3) in the form of (6) it suffices to introduce

$$\tilde{T}_{Rij} \equiv 2\frac{q^2}{R^6} W^2 k_i k_j \tag{7}$$

and

$$A_{ij} \equiv \frac{q^2}{R^4}\left(a^r a_r - 3\frac{W^2}{R^2}\right) k_i k_j \tag{8}$$

It is quite simple to check that the two parts in which the energy-momentum tensor are split are dynamically independent in Teitelboim's sense [2], that is, the divergence of each term vanishes outside of the world line

$$\tilde{T}_R{}^{ib}{}_{,b} = A^{ib}{}_{,b} = 0 \tag{9}$$

and that

$$\tilde{T}_R{}^{ib} k_b = A^{ib} k_b = 0 \tag{10}$$

so that (7) and (8) are true radiation tensors [6] not modifying the radiation rate at infinity.

We can now use pre-existing expressions that were calculated, for example, in [3] or [5] to evaluate the linear and angular momentum fluxes through the particle's light cone (the surface of constant proper time, τ) and through the surface of constant retarded distance (i.e. constant R)—that is, through the walls of a Bhabha–Synge tube—contributed by A_{ij}. In this way we obtain

$$\begin{aligned}
&\int_{R=\text{constant}} A^{ib}\,\mathrm{d}\sigma_b = R^2\int_{\tau_1}^{\tau_2}\mathrm{d}\tau\int A^{ib}R_{,b}\,\mathrm{d}\Omega = 0\\
&\int_{\tau=\text{constant}} A^{ib}\,\mathrm{d}\sigma_b = -\int_{R_1}^{R_2} R\,\mathrm{d}R\int A^{ib}k_b\,\mathrm{d}\Omega = 0\\
&\int_{R=\text{constant}} M^{ijb}\,\mathrm{d}\sigma_b = \int_{\tau=\text{constant}} M^{ijb}\,\mathrm{d}\sigma_b = 0
\end{aligned} \tag{11}$$

where $M_{ijr} \equiv X_i A_{jr} - X_j A_{ir}$, X_i an event in Minkowski space, is the angular momentum density associated with (8). These results justify our claim that, with regard to energy-momentum flows through a Bhabha–Synge tube, A^{ij} does not contribute and then the modified Maxwell tensor $\tilde{T}_R^{ij}$ is equivalent to expression (3) for the radiative Maxwell tensor.

3. The superpotential for the radiative part

Let us exhibit that the whole radiative part of the Maxwell tensor of a LWF can be obtained from a world-line-dependent superpotential. As electromagnetic radiation is obviously a non-local phenomenon there can be no surprise in finding that the superpotential is given in terms of integrals over the world line of the radiating charge. We must point out, however, that this is not the first time a non-local superpotential has been proposed, see for example [14].

It is well known that the radiative, T_R^{ab}, and the bounded, T_B^{ab}, parts of the energy-momentum tensor of a LWF are dynamically independent for, outside of the charge world line, we have

$$\begin{aligned} T_{B}{}^{ib}_{,b} &= 0 \\ T_{R}{}^{ib}_{,b} &= 0. \end{aligned} \tag{12}$$

Some years ago Van Weert [12] showed that the first of equations (12) maybe regarded as a consequence of the existence of a superpotential $K_B{}^{ijr} = -K_B{}^{jir}$ generating the bounded part of the Maxwell tensor through

$$T_B{}^{cr} = K_B{}^{rjc}{}_{,j}. \tag{13}$$

The Van Weert superpotential $K_B{}^{ijr}$ can be interpreted as an intrinsic angular momentum (or spin) density for the field of a moving charge [14]. This has been established by noticing that the algebraic and differential properties of $K_B{}^{ijr}$ happen to be identical to the properties of the Lanczos superpotential in general relativity [15, 16], this is an interesting property which permits analogies between LWF in classical electrodynamics and Robinson–Trautman solutions in general relativity. Using such analogies we have been able to introduce a conformal tensor for the radiative part of a LWF and have managed to classify algebraically the LWF [17] as a Petrov-type II field in the generic case (note that the expression for the symmetric Maxwell tensor is incorrectly written in that paper).

Now we follow Van Weert's idea—though not its procedure since he did not say how he got its superpotential—to show that the last of equations (12) can be regarded as a consequence of the existence of a superpotential with the symmetry property $K_R{}^{ijr} = -K_R{}^{jir}$ but that happens to be non-local. This quantity $K_R{}^{ijr}$ is such that

$$T_R{}^{bc} = K_R{}^{c}{}_{j}{}^{b,j}. \tag{14}$$

The problem is, as we think happened with Van Weert's derivation, we cannot give a systematic procedure for obtaining the superpotential, we can only state the equations that, in a very informal trial and error fashion, guided us to the expression for $K_R{}^{abc}$.

To begin with, let us consider the expression for the radiative part of the energy-momentum tensor (3) together with Maxwell equations outside of the world line

$$\begin{aligned} &F^{ab}, b = 0 \\ &{}^*F^{ab}, b = 0 \qquad (\text{where } {}^*F^{ab} \equiv \epsilon^{ab}{}_{cd} F^{cd}/2 \text{ is the dual of } F^{ab}) \end{aligned} \tag{15}$$

and with the fact that the null four-vector k^a is an eigenvector of the electromagnetic field F^{ab} [17]:

$$F^{ab} k_b = \frac{q}{R^2} k^a. \tag{16}$$

It is also useful to introduce an orthonormal tetrad $e^c_{(a)}$ $(a = 1, 2, 3, 4)$ on the charge's world line, defined in such a way that $e^r_{(4)} \equiv v^r$ and such that its unit space-like part evolves according to the law [18]

$$\frac{\mathrm{d}}{\mathrm{d}\tau} e^r_{(\gamma)} = a_{(\gamma)} v^r \qquad \gamma = 1, 2, 3 \tag{17}$$

where the $a_{(\gamma)} \equiv a^b e_{(\gamma)b}$ are the projections of the four-acceleration a^r over the spatial triad $e^r_{(\gamma)}$, with $\gamma = 1$, 2, 3. Guided by equations already mentioned we have managed to come across the following expression for the potential which generates the radiative part of the Maxwell tensor of a LWF:

$$\begin{aligned} K_R{}^{bjc} = \frac{q^2}{4R^2} \Bigg[\left(a^c + \frac{4}{R^2} W k^c \right) (a^b k^j - a^j k^b) \\ + \frac{3W^2}{R^2} (\eta^{cb} k^j - \eta^{cj} k^b) - \frac{W}{R} (a^c + W k^c / R^2)(v^b k^j - v^j k^b) \Bigg] \\ - 2q F^{bj} p_{(\sigma)} p_{(\theta)} \int_0^\tau a_{(\sigma)}(s) a_{(\theta)}(s) v^c(s) \, \mathrm{d}s \end{aligned} \tag{18}$$

where we sum over $\sigma, \theta = 1, 2, 4$; $F^{bj} = q(U^b k^j - U^j k^b)/R^2$ with $U^r \equiv B v^r + a^r$, is the full LWF and we are employing the quantity $P_{(\sigma)} \equiv p_r e^r_{(\sigma)}$, this is the projection on the tetrad of the quite useful four-vector [3, 13]

$$p^q \equiv \frac{k^a}{R} - v^a. \tag{19}$$

Notice that, as we might expect, expression (18) vanishes with the four-acceleration. The whole energy-momentum tensor of a LWF can be obtained thus as the four-divergence

$$T^{cr} = (K_B^{rjc} + K_R^{rjc})_{,j} \tag{20}$$

as can be easily shown by computing the Maxwell tensor using equation (18). In fact this is also the simplest way to check our expression for the new superpotential.

Quantities defined in a similar way to the non-local part of the superpotential $K_R{}^{bjc}$ are rather useful in classical electrodynamics. For example, let us introduce the non-local and non-symmetric tensor

$$P_{ab}(x_j) \equiv \frac{2q^2}{3} U_a R \int_{-\infty}^\tau a^r(s) a_r(s) v_b(s) \, \mathrm{d}s \tag{21}$$

where x_j is the position of the charge when the proper time is τ. The integral can be shown to be finite, at least is we accept that the motion is asymptotically uniform (i.e. uniform as $\tau \to -\infty$). We can easily show now that this non-local tensor is related with the moving charge's radiation rate at infinity, i.e. with the Larmor formula, through

$$P_{ib}{}^{,i} = \frac{2q^2}{3} (R^{,i} U_i + R U_i{}^{,i}) \int_{-\infty}^\tau a^2(s) v_b(s) \, \mathrm{d}s - \frac{2q^2}{3} U_i a^2 v_b k^i = \frac{2q^2}{3} a^2 v_b \tag{22}$$

where we have used that $\tau_{,b} = -k_b/R$ and $a^2 \equiv a^r a_r$.

As another example of the usefulness of non-local tensors, let us define the tensor

$$L_{ab} \equiv q U_a \int_{-\infty}^\tau v_b(s) \, \mathrm{d}s \tag{23}$$

which, in this case, is only apparently non-local as it trivially becomes the local quantity $q U_a (x_b(\tau) - x_b(-\infty))$, where $x^a(s)$ stands for the charge's position on the world line at the proper time s. Using the basic properties of the Synge vector given above, this tensor, or the local quantity to which it reduces, can be shown to generate the Liénard–Wiechert electromagnetic four-potential

$$L^{a}{}_{b,a} = qU_a k_b \left(-\frac{k^a}{R}\right) = q\frac{k_b}{R} = A_b. \tag{24}$$

It should be clear by now that L^{ab} can also be useful when obtaining a superpotential for the Faraday tensor $F_{ab} \equiv A_{b,a} - A_{a,b} = L_{rb,ra} - L_{ra,rb} \equiv S_{bra,r}$, where the superpotential obviously should be defined as

$$S_{sra} \equiv L_{as,r} - L_{ar,s}. \tag{25}$$

4. Concluding remarks

We have shown how the radiative part of an energy-momentum tensor of the Liénard–Wiechert field can be expressed as the sum of an inactive part not contributing to the energy-momentum balance plus an active part solely responsible for the balance. This splitting has been found to be useful for removing the classical renormalization procedure thought to be neccesary for obtaining the Lorentz–Dirac equation [13, 19]. We have shown how to obtain the radiative part of the tensor as the four-divergence of a non-local superpotential as corresponds to a quantity related to electromagnetic radiation. This superpotential still lacks an appropriate physical interpretation. We have also illustrated the relevance of certain non-local tensors for classical electrodynamics by defining tensors related, one with the Larmor formula, and the other with the Liénard–Wiechert electromagnetic potential.

As, perhaps, we have managed to make apparent in this work, classical electrodynamics is still a fascinating theoretical subject, in which lots of things have yet to be done. Even some details associated with the most basic electrodynamic fact of all, the electromagnetic field produced by a classical charged particle in motion, are not yet fully understood, many of its basic problems remain unsolved [19–21]. Although this work only touches tangentially, if at all, the basic problems associated with the motion of charged particles [19] we, nevertheless, think the results obtained here might be useful in its study and in the analysis of the theoretical structure of classical electrodynamics [13, 14, 17, 19–23].

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